Traces via Strategies

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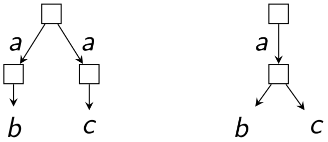

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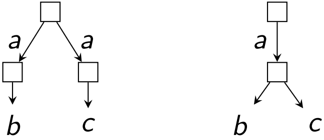

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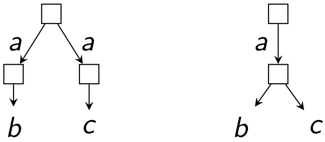

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- Finite trace semantics have enjoyed a general coalgebraic treatment in [Hasuo et al., 2007], which covers LTSs, MCs, CFGs.

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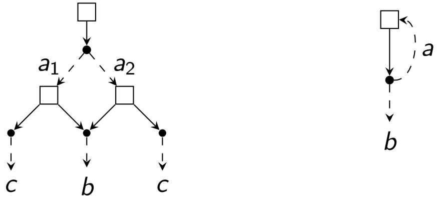

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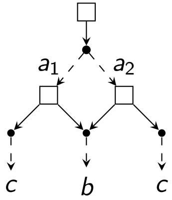

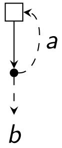

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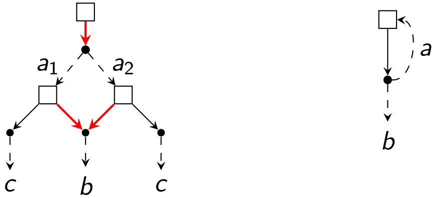

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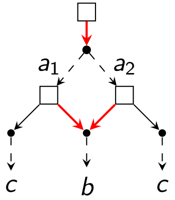

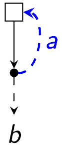

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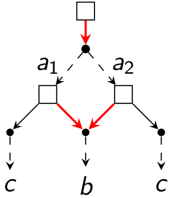

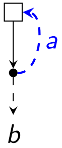

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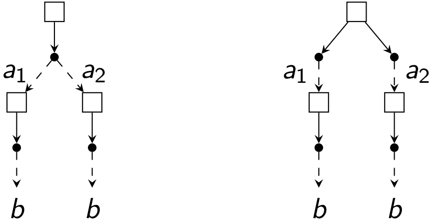
- A Markov decision process is a also a type of game, where the environment probabilistically updates.

When are games trace equivalent?



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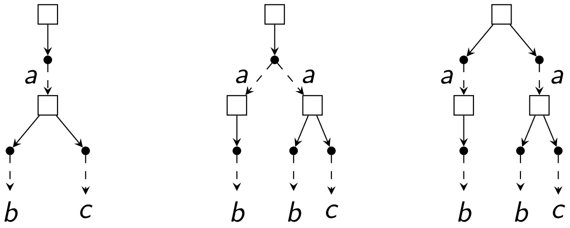

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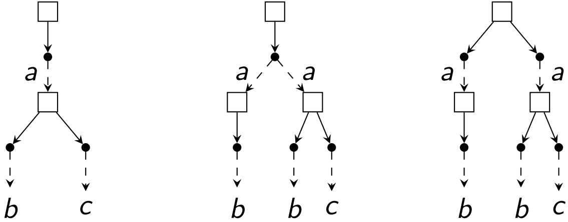

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- If so, what are the trace semantics at a state in a game?

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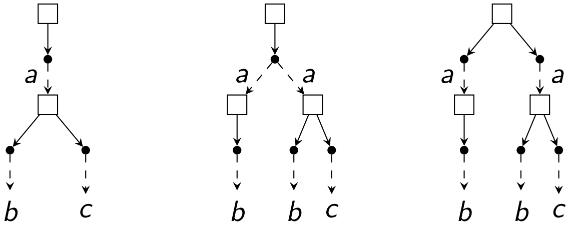

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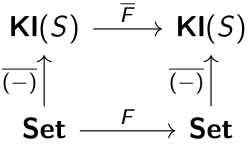
Goal: phrase synthesis coalgebraically to obtain general theory.

Outline

1. Choose a monad S and functor F to model two-player games, giving us a finite trace semantics map [Hasuo et al., 2007]:

$$\mathrm{tr} : \mathrm{States} \to S(\mathrm{Initial}_F)$$

2. Give a coalgebraic/categorical definition of strategies.
3. Characterising the trace map.

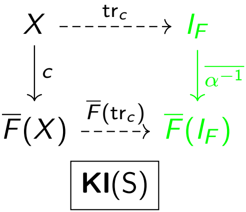


Theorem ([Hasuo et al., 2007])

- *A monad S : **Set** $\to$ **Set** such that*
  1. $\mathbf{Kl}(S)$ *is* ω-**cpo** *enriched*
  2. $S(0) \cong 1$
  3. *Zero maps* $X \to 0 \to Y$ *are bottom elements*
- *A functor F : **Set** $\to$ **Set** such that*
  1. *F lifts to* $\overline{F} : \mathbf{Kl}(S) \to \mathbf{Kl}(S)$
  2. $\overline{F}$ *is locally monotone*
  3. *The initial F-algebra* $\alpha : F(I) \to I$ *is*

     $\mathrm{colim}(0 \xrightarrow{!} F(0) \xrightarrow{F(!)} F^2(0) \xrightarrow{F^2(!)} \cdots)$

Then $\overline{\alpha^{-1}} : I_F \to \overline{F}(I_F)$ *is the final* $\overline{F}$*-coalgebra.*



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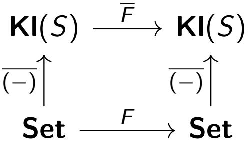
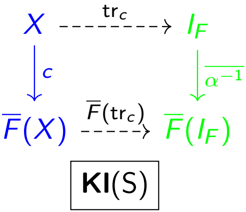
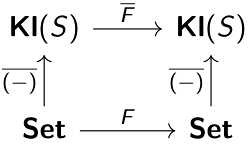




This is Theorem 3.3 of [Hasuo et al., 2007], presented with slightly weaker assumptions like in [Jacobs, 2016]



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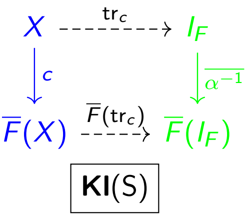

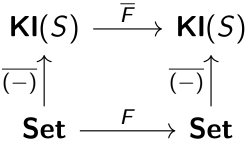

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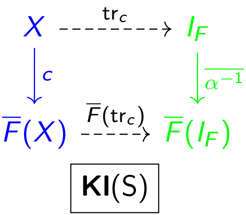

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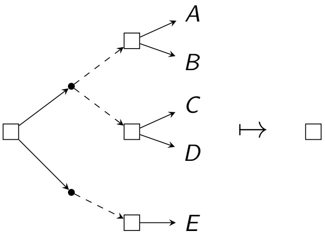

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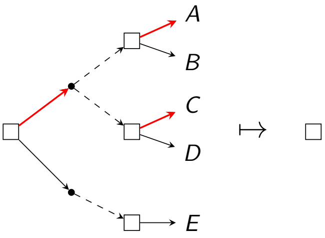

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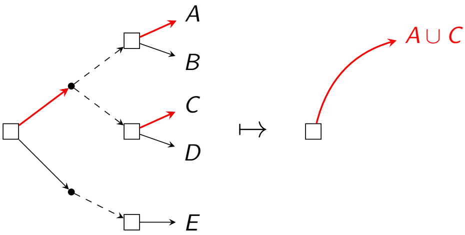

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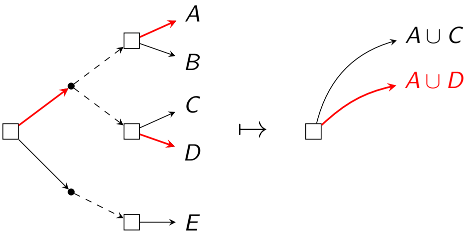

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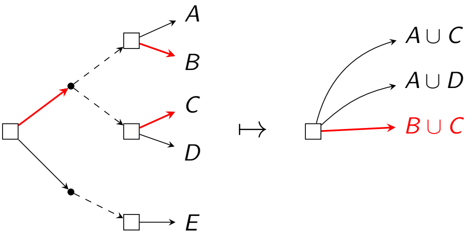

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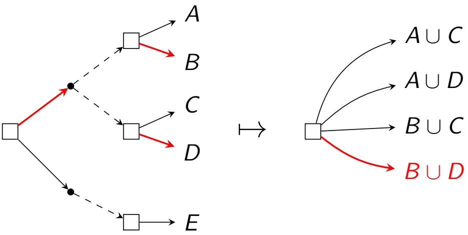

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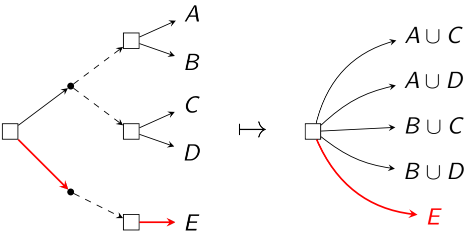

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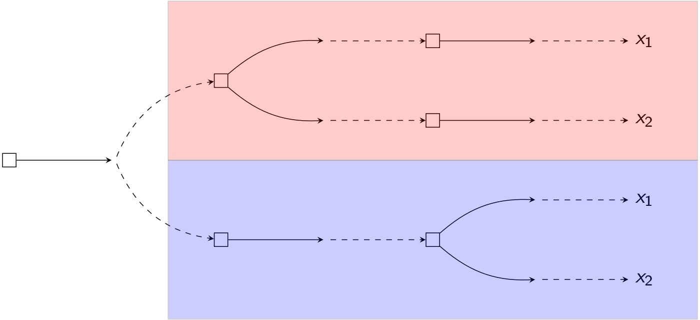

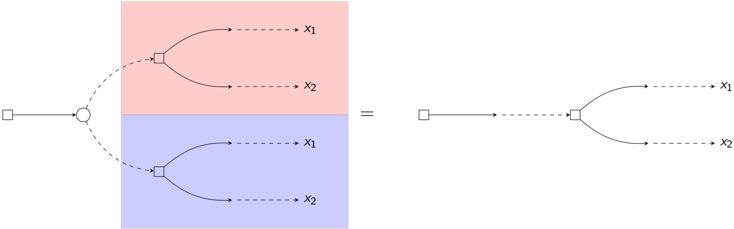

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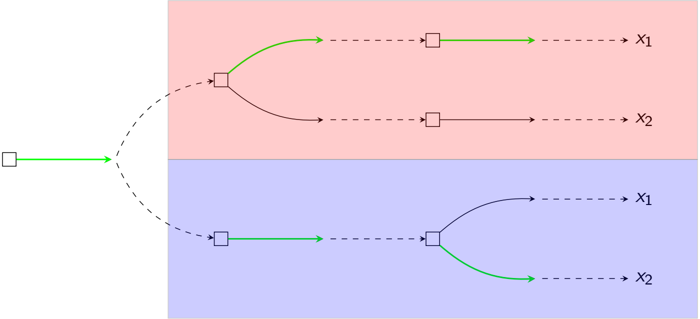

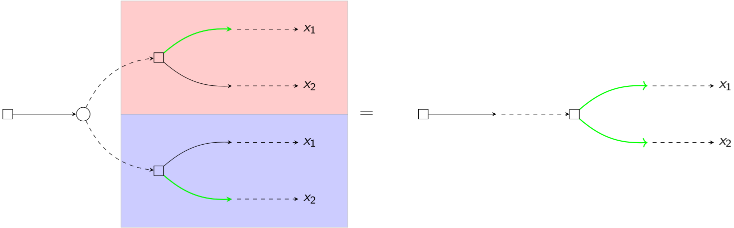

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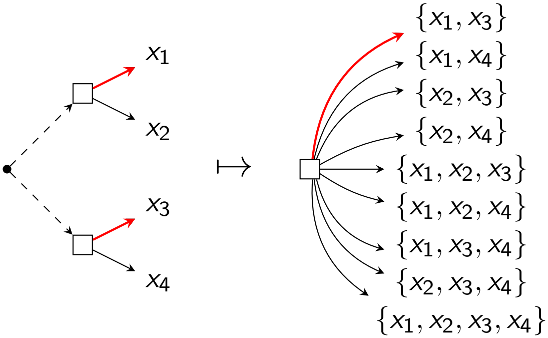

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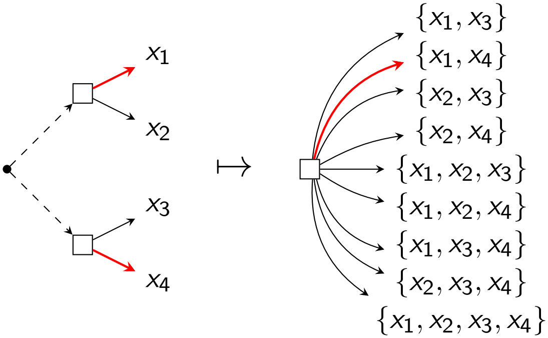

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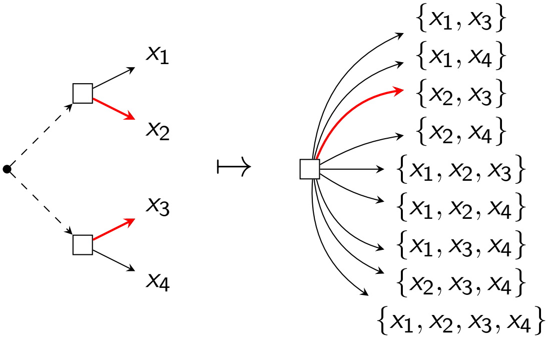

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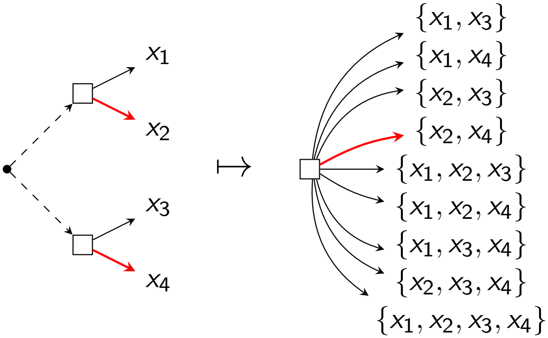

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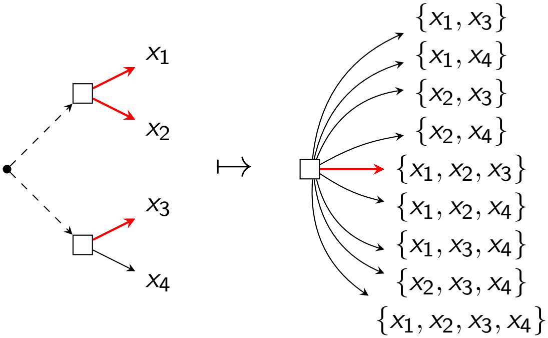

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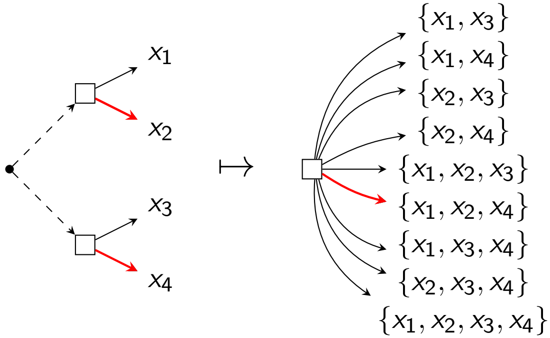

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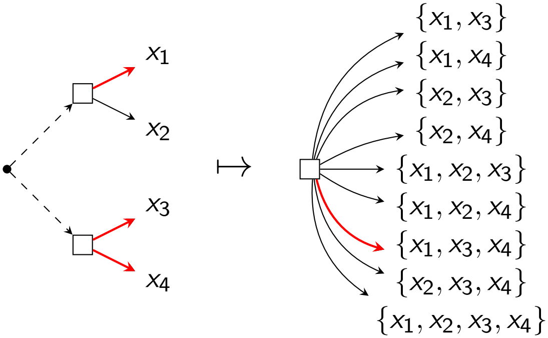

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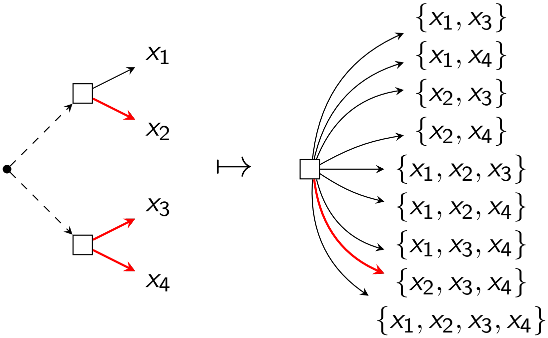

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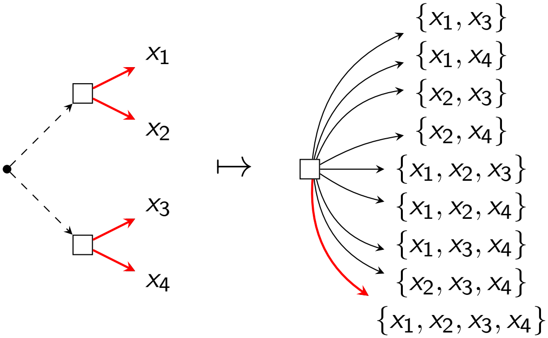

This law was used in [Goy, 2021] to model alternating automata, see [Garner, 2020] for composite monad construction

The law $DP \to PD$ gives a monad $\widetilde{PD} : \mathbf{Set} \to \mathbf{Set}$

- Let D be the finite distribution monad.
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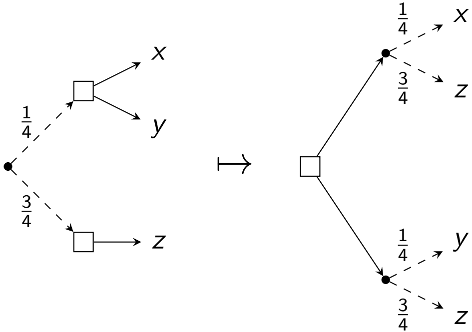

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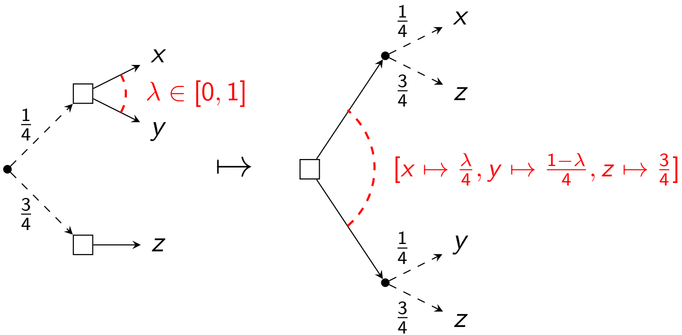

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Theorem ([Hasuo et al., 2007])

- *A monad $S : \mathbf{Set} \to \mathbf{Set}$ such that*
  1. $\mathbf{Kl}(S)$ *is* ω-**cpo** *enriched*
  2. $S(0) \cong 1$
  3. *Zero maps* $X \to 0 \to Y$ *are bottom elements*
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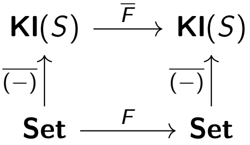
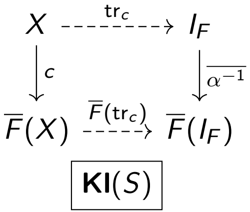
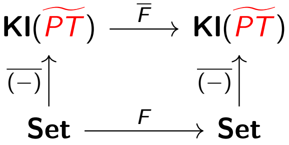
Then $\overline{\alpha^{-1}} : I_F \to \overline{F}(I_F)$ *is the final* $\overline{F}$*-coalgebra.*







Theorem ([Hasuo et al., 2007])

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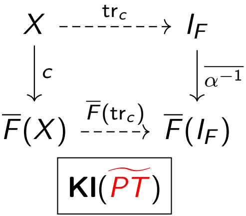

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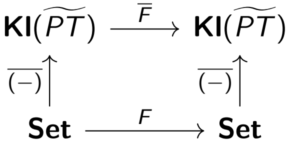

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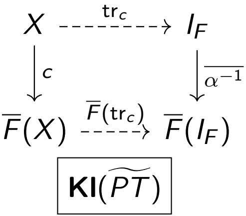

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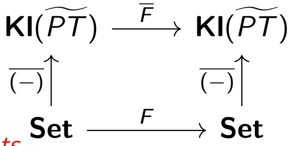

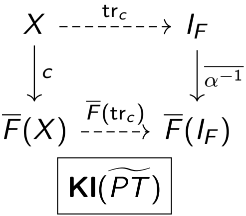

$PP(0) \cong P(1) \cong 2$

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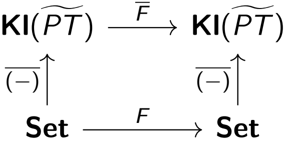

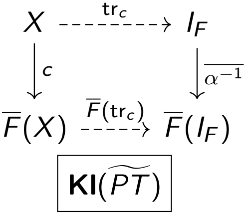

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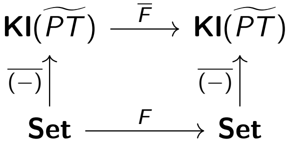

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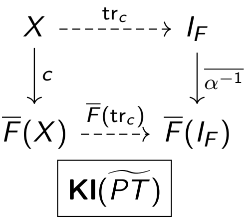

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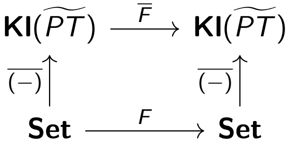

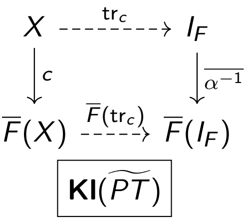

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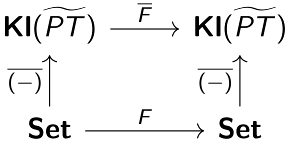

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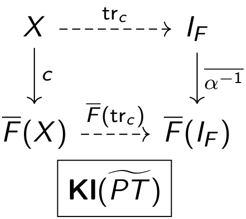

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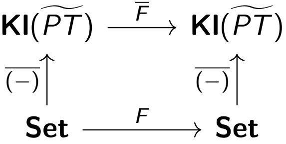

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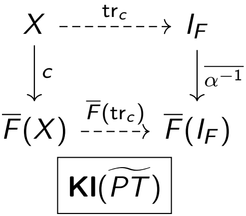

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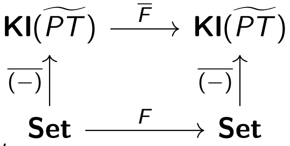

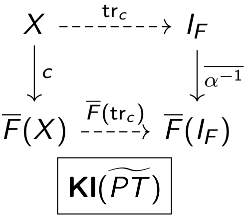

$\widetilde{PT}$ is commutative $\implies$ F lifts to $\overline{F}$

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$\widetilde{PT}$ is not commutative

$$
\begin{array}{ccccc}
S(SX \times Y) & \xleftarrow{\text{stl}} & SX \times SY & \xrightarrow{\text{str}} & S(X \times SY) \\
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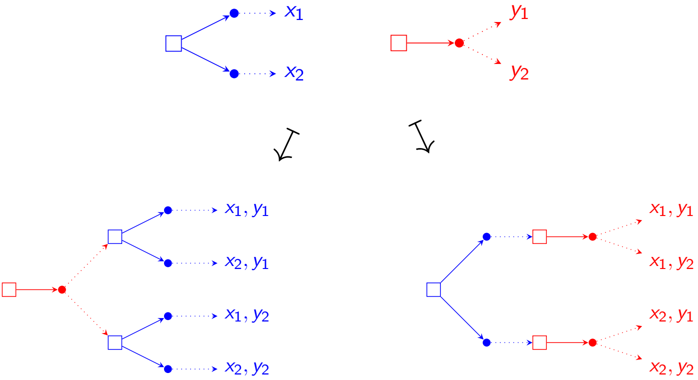

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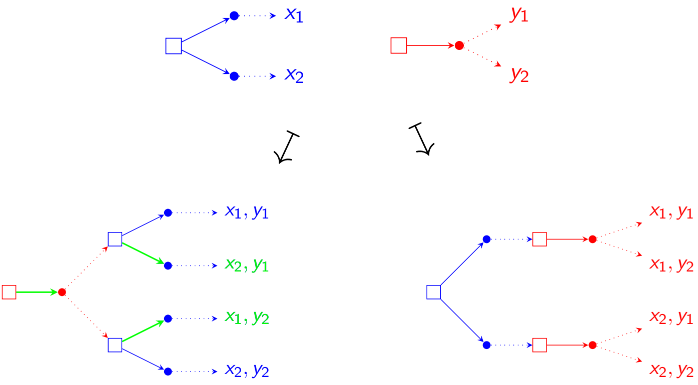

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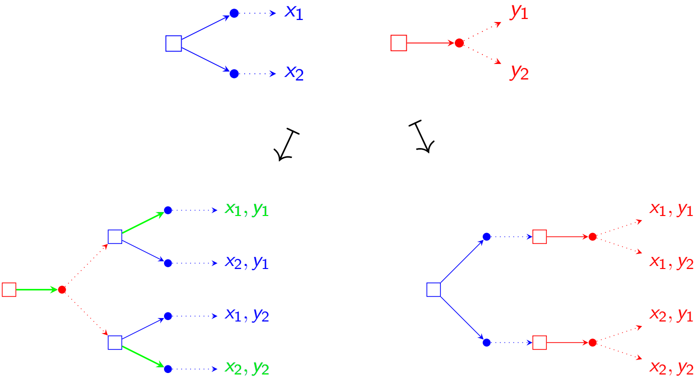

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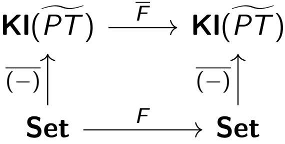

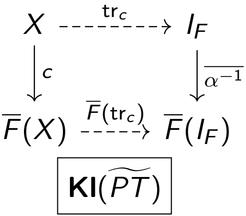

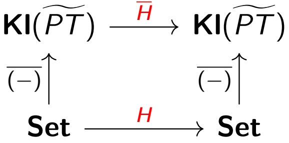

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- *Let $H : \mathbf{Set} \to \mathbf{Set}$ be $B + A \times (-)$*
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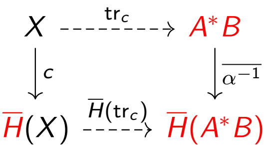

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$\mathbf{Kl}(\widetilde{PT})$

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$$\begin{array}{ccc} X & \overset{\mathrm{tr}_c}{\dashrightarrow} & A^*B \\ \downarrow c & & \downarrow \overline{\alpha^{-1}} \\ \overline{H}(X) & \overset{\overline{H}(\mathrm{tr}_c)}{\dashrightarrow} & \overline{H}(A^*B) \end{array}$$

$\boxed{\mathbf{Kl}(\widetilde{PT})}$

Games live in $\mathbf{Kl}(\widetilde{PT})$

$$c : X \to \widetilde{PT}H(X)$$

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- Both are morphisms $X \to \overline{H}(X)$ in $\mathbf{Kl}(\widetilde{PT})$
- Where do strategies live?
  - A strategy picks some $TH(X)$ in $c(x)$
  - Strategies live in $\mathbf{Kl}(T)$

Categorical definition of a strategy

- Strategies depend on past states, as well as behaviour.

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Definition

A strategy is a chain of maps $\{\sigma_n\}_{n \in \omega}$

- $\sigma_0 : 1 \to TX$ picks an initial state
- $\sigma_{n+1} : \mathrm{Im}(\sigma_n) \to TH_X^{n+1}(X)$ extends plays

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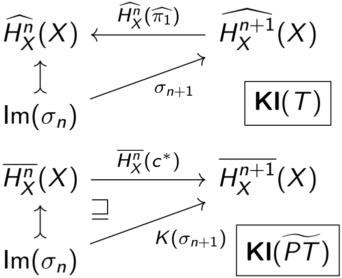
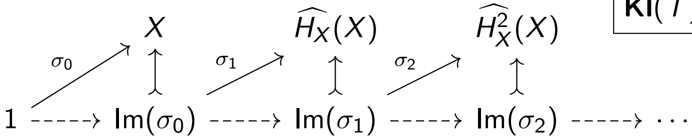


Strategy outcomes

KI(T)



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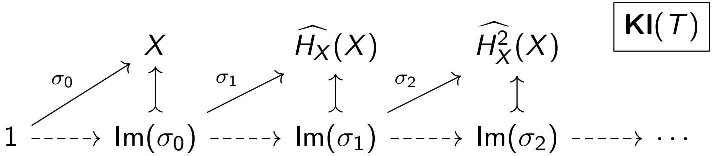

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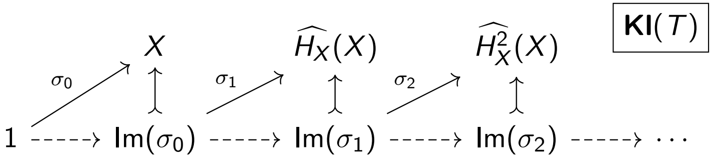

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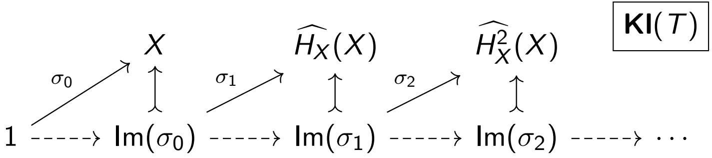
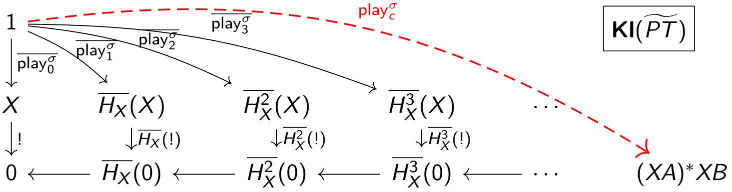


- Composing along the bottom and then up we can obtain an *n-step partial outcome* $\mathrm{plays}^\sigma_n : 1 \to TH^n_X(X)$
- For a full outcome, we want map into $(XA)^*XB$
- We know $(XA)^*XB$ is the limit of the final sequence in $\mathbf{Kl}(\widetilde{PT})$



What is in the trace map?

$$\begin{array}{ccc} X & \overset{\mathrm{tr}_c}{\dashrightarrow} & A^*B \\ \downarrow c & & \downarrow \overline{\alpha^{-1}} \\ \overline{H}(X) & \overset{\overline{H}(\mathrm{tr}_c)}{\dashrightarrow} & \overline{H}(A^*B) \end{array}$$

- $\mathrm{tr}_c : X \to \widetilde{PT}(A^*B)$
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Theorem

The trace semantics at x are the union of outcomes of strategies starting in x

$$\mathrm{tr}_c(x) = \bigcup_{\sigma \text{ starts in } x} \mathrm{tr}_c^\sigma$$

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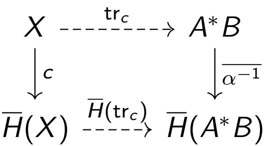

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Unfolding the game n times from x, is equal to the n-step partial traces of n-step strategies starting in x

$$H^{n-1}(c) \circ \cdots \circ H(c) \circ c(x) = \{\mathrm{tr}_n^\sigma \mid \{\sigma_i\}_{i \leq n} \text{ starts in } x\}$$

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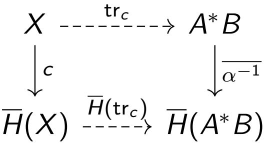

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Theorem

The trace semantics at x are the union of outcomes of strategies starting in x For any $U \in T(A^*B)$:

$$U \in \mathrm{tr}_c(x) \iff \text{there is a strategy which forces } U$$

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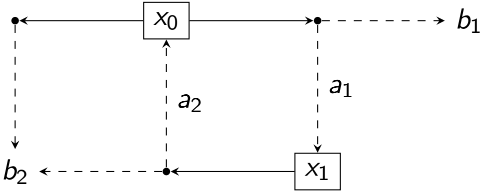
Example: $U \in \mathrm{tr}_c(x_0) \iff$ a strategy forces U



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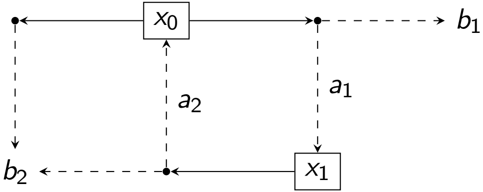

$\{b_1\}$

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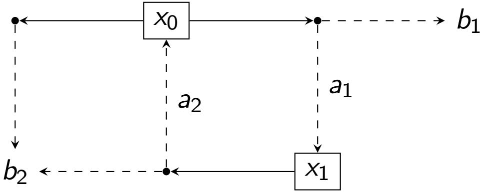

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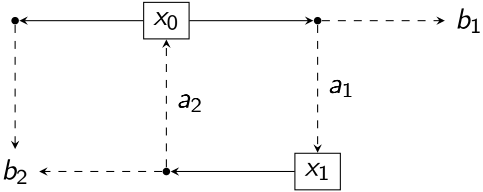
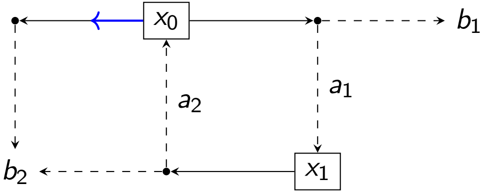


$\{b_1\}$ No

$\{b_2\}$

Example: $U \in \mathrm{tr}_c(x_0) \iff$ a strategy forces U



$\{b_1\}$	No
$\{b_2\}$	Yes

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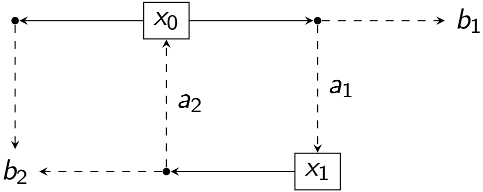

$\{b_1\}$	No
$\{b_2\}$	Yes
$\{a_1 a_2 b_2, a_1 b_2, b_1\}$	

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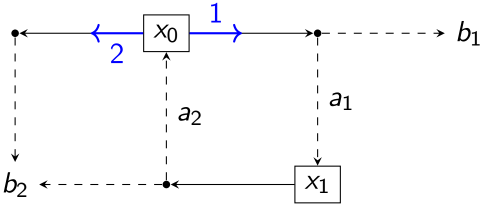

$\{b_1\}$	No
$\{b_2\}$	Yes
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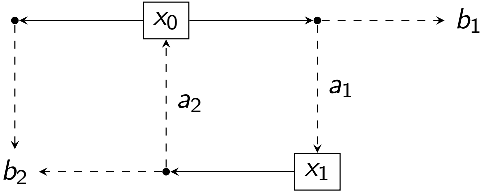

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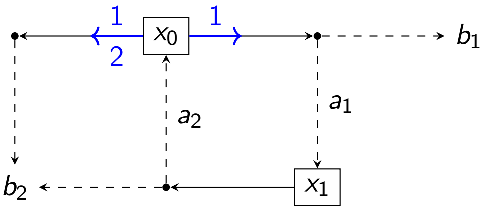

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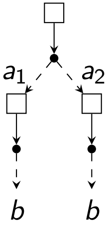
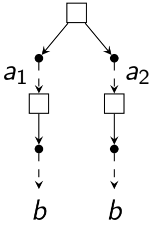
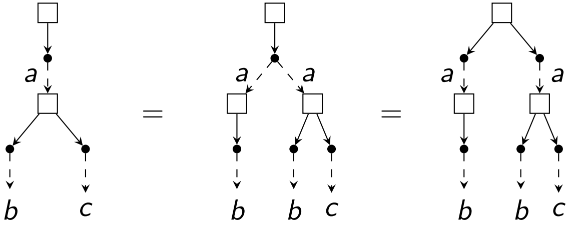
$\{a_1 a_2 b_2, a_1 b_2, b_1, b_2\}$	Yes



$\subseteq$

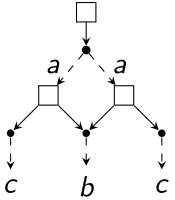


$\{\{a_1 b, a_2 b\}\}$ $\{\{a_1 b\}, \{a_2 b\}, \{a_1 b, a_2 b\}\}$

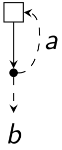


$$\{\{ab\}, \{ac\}, \{ab, ac\}\}$$

Conclusion



$\{\{ab\}, \{ac\}, \{ab, ac\}\}$



$\emptyset$

This talk:

- Obtained a finite trace semantics map
- Defined strategies in $\mathbf{Kl}(T)$
- Characterised the trace map in terms of strategies

Future work

- Product construction with an automaton
- Infinite traces

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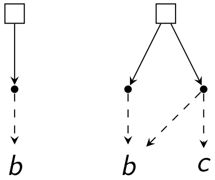
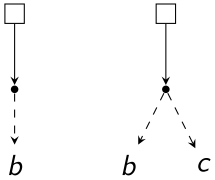

b
b
c


b
b
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