A Coalgebraic Approach to Infinite Games

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Why infinite games?

- Verification: does an implementation exhibit some desired behavour φ?
  - Implementation typically modelled as a *transition system*
  - Two (dual) flavours:
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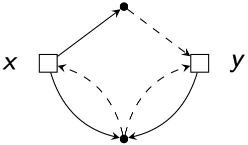

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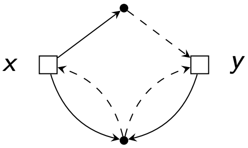

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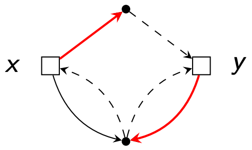

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- Given a type of arena (e.g. MDPs) and objective e.g. (reachability, ω-regular):
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- We are looking for a mathematical framework for games
  - *Parametric* in the type of arena/objective
  - Benefits: gain insight into limitations, generic solutions to general classes of games, transfer of results and algorithms

Our coalgebraic approach

- First point of call: *coalgebra*
- Coalgebraic techniques have already given us generic methods in automata theory:
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- Initial steps towards treating *finite* two-player games coalgebraically [Plummer and Cirstea, 2025]:
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- We develop, in *infinite games*, the coalgebraic theory of:
  - Strategy outcomes: the set of plays which conform to a strategy
  - Game outcomes: the set of strategies in the game
    - Game outcome at a state x: set of strategies starting from x

The game arena as a coalgebra

- We are concerned with the game arena
  - I.E. a coalgebra $\delta : X \to PP(X)$
  - Each $U \in \delta(x)$ is controller move
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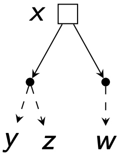

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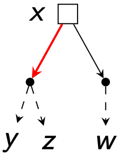

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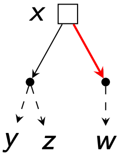

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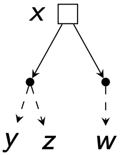

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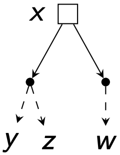

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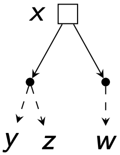

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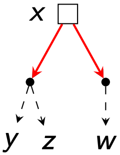

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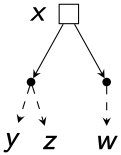

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  - There is a monad structure on $\mathcal{G}$

Strategies as a chain of relations

- A relation $f : X \nrightarrow Y$ is
  - *right-total* when every point y has some x with $y \in f(x)$
  - *separating* when $f(x) \cap f(x') = \emptyset$ for disinct x, x'
  - separating and right-total iff $f^\dagger$ is a function $Y \to X$

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- A strategy is a chain of right-total relations:

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  - such that:
    - $\sigma_n(x_0 \ldots x_n)$ comes from a choice in the game
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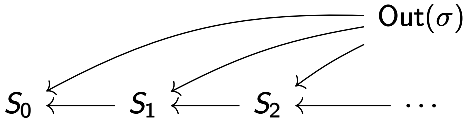

Strategy outcomes: alternative characterisations

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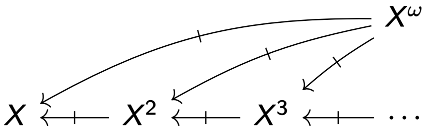

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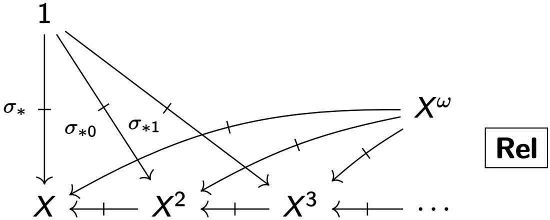

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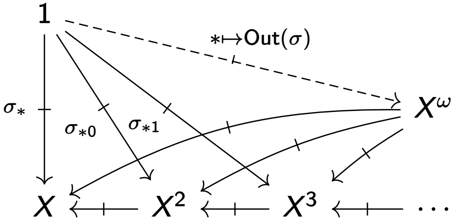

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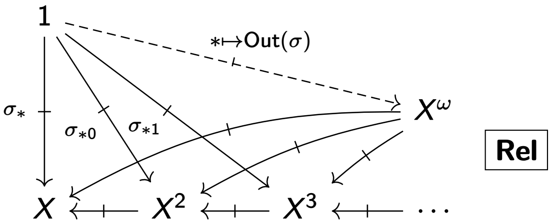
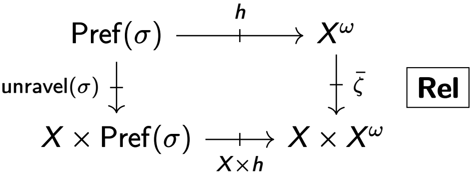


Largest homomorphism:



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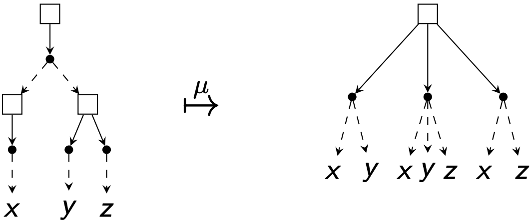

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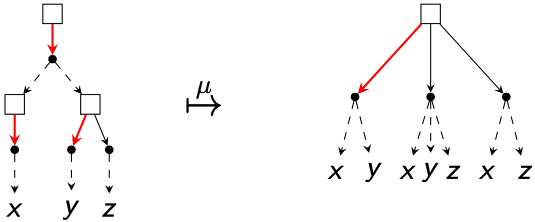

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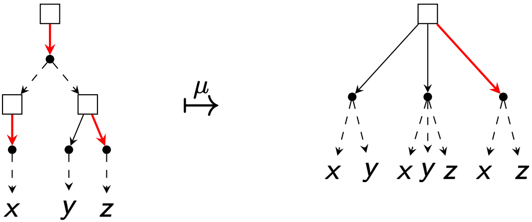

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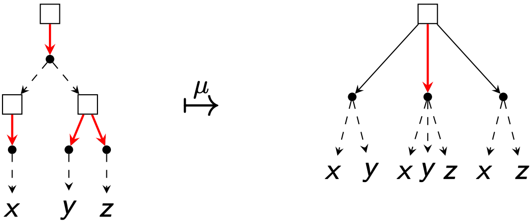

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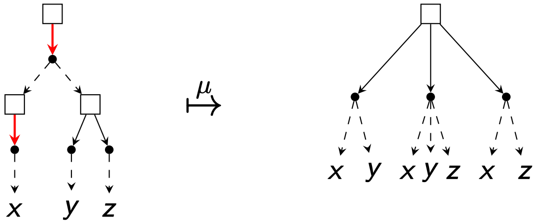

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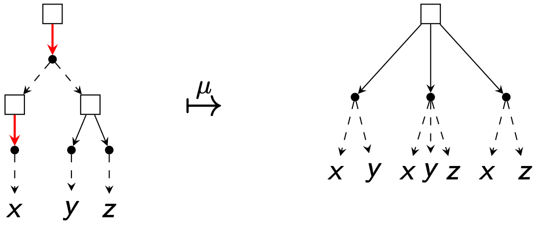

- Given a game $\delta : X \to \mathcal{G}(X)$
- We use a $\gamma : X \to \mathcal{G}(X \times X)$ which *records state information*:

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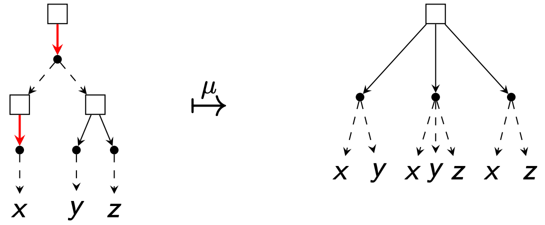

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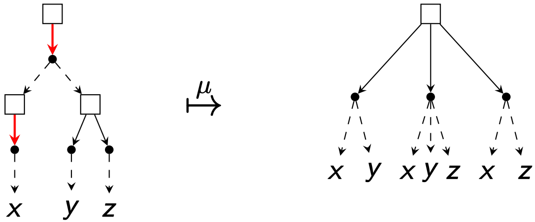

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  - We can show $\gamma_n(x)$ consists of n-step strategies starting from x

The game outcome as a limit

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- We can define the outcome of γ as a categorical limit:

$$\mathrm{Im}(\gamma_0) \longleftarrow \mathrm{Im}(\gamma_1) \longleftarrow \mathrm{Im}(\gamma_2) \longleftarrow \cdots$$

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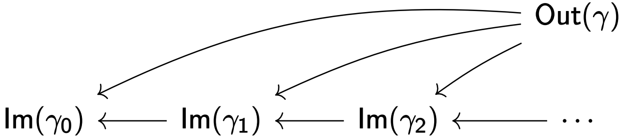

Alternative characterisation (1): greatest homomorphism

- There exists a greatest homomorphism $f : X \to \mathcal{G}(X^\omega)$:

$$
\begin{array}{ccc}
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{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\bar{\bar{\zeta}}} \\
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\end{array}
\qquad \boxed{\mathbf{Kl}(\mathcal{G})}
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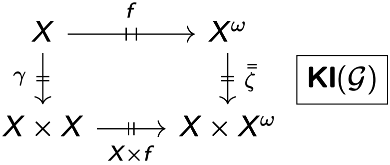

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- It is possible to restrict homomorphisms such that each $V \in f(x)$ is *limit closed* [Emerson, 1983]
  - Roughly means V contains every possible infinite play which conforms to finite approximations of V
- Theorem: the greatest homomorphism f which contains limit-closed subsets recovers the outcome of a game.
  - $f(x)$ gives us the strategies which start in x

$$f(x) = \mathrm{Out}(\gamma)(x)$$

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- The least such map $\perp_\gamma$ maps $x \mapsto \{Z_{\gamma,x}\}$
- Theorem: let $f : X \to \mathcal{G}(X^\omega)$ be the least homomorphism $(X, \gamma) \to (Z, \overline{\overline{\zeta}})$ such that $Z_{\gamma,x} \in f(x)$, if we close $f(x)$ under intersections of descending chains of subsets, we obtain the map which assigns states to their outcomes.

$$\text{close}(f(x)) = \text{Out}(\gamma)(x)$$

Conclusion

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- Future work:
  - Expand the picture to:
    - Stochastic environments (MDPs)
    - Stochastic + nondeterministic environments (SSGs)
    - Weighted transitions (energy games)
  - Add objectives:
    - Coalgebraic approach to Büchi/parity conditions [Urabe et al., 2016]
    - Product construction of a game with an automaton
  - Generic techniques for computing/approximating solutions

References I

Emerson, E. (1983).
Alternative semantics for temporal logics.
Theoretical Computer Science, 26(1):121–130.

Plummer, B. and Cirstea, C. (2025).
Traces via strategies in two-player games.
Electronic Notes in Theoretical Informatics and Computer Science, Volume 5 - Proceedings of MFPS XLI.

Urabe, N., Shimizu, S., and Hasuo, I. (2016).
Coalgebraic Trace Semantics for Buechi and Parity Automata.
In Desharnais, J. and Jagadeesan, R., editors, *27th International Conference on Concurrency Theory (CONCUR 2016)*, volume 59 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 24:1–24:15, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.