

# Infinite Traces in Strategies and Games

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University of Southampton

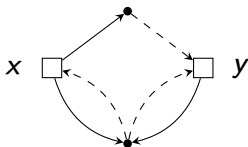
April, 2026

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- ▶ Two players: controller ( $\square$ ) versus environment ( $\bullet$ )

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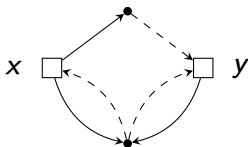
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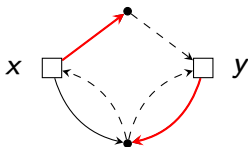


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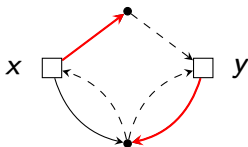


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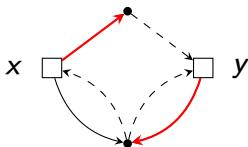


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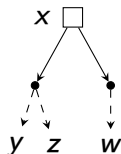
- ▶ Is there a winning strategy for  $\square$ ?
  - ▶ There are infinitely many: strategies are history dependent
- ▶ Terminology:
  - ▶ *Strategy outcome*: the set of execution paths which conform to it
  - ▶ *Game outcome* at  $x$ : the set of strategies which start in  $x$ 
    - ▶ Could be called the “strategy semantics”

## ... as a coalgebra

- ▶ We are concerned with the game arena
  - ▶ I.E. a *PP*-coalgebra  $\delta : X \rightarrow PP(X)$
  - ▶ Each  $U \in \delta(x)$  is controller move
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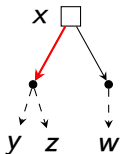
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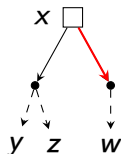
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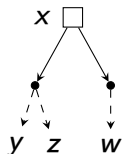
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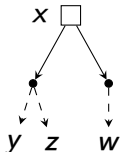


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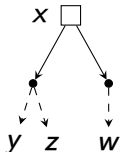


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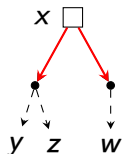


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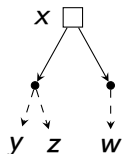


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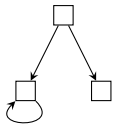
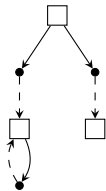


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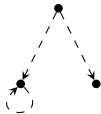
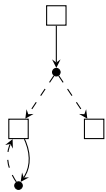
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  - ▶ There is a monad structure on  $\mathcal{G}$

# Strategies generalise executions paths and transition systems

$$P(X) \xrightarrow{P(\eta^P)} PP(X)$$

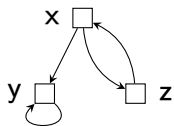

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- ▶ Given a transition system, compute the execution paths originating from each state



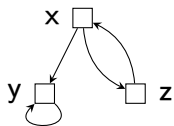
$$x \mapsto x(zx)^*(y^\omega + (zx)^\omega)$$

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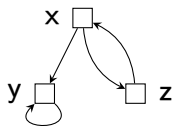
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- ▶ Work in **Rel**, the category of sets and relations
  - ▶ This moves from branching-time to linear-time semantics
  - ▶  $\overline{(-)} : \mathbf{Set} \rightarrow \mathbf{Rel}$  lifts functions into relations
  - ▶ Use  $\rightarrow$  to denote relations
  - ▶ **Rel** is **Pos**-enriched with subset inclusion

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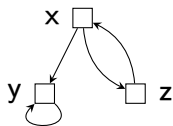
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- ▶ Model a transition system  $\delta : X \rightarrow X$  as an endorelation
- ▶ Let  $F := X \times (-) : \mathbf{Set} \rightarrow \mathbf{Set}$ 
  - ▶ The terminal  $F$ -coalgebra  $(Z, \zeta)$  is  $(X^\omega, \langle \text{hd}, \text{tail} \rangle)$
  - ▶  $F$  lifts to  $\overline{F} : \mathbf{Rel} \rightarrow \mathbf{Rel}$
  - ▶ Use  $\gamma : X \rightarrow F(X)$  to record state information:
  - ▶  $\gamma(x) := \{(x, x') \mid x' \in \delta(x)\}$

## Look at $\bar{F}$ -coalgebra morphisms into $(Z, \bar{\zeta})$

- ▶ Use  $\gamma : X \rightarrow F(X)$  built from  $\delta : X \rightarrow X$  from the previous slide

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Z \ (\cong X^\omega) \\
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- ▶ Is  $(Z, \bar{\zeta})$  a terminal  $\bar{F}$ -coalgebra?
  - ▶ Is  $f$  *uniquely determined* by the above property?

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- ▶ The solution in [Jacobs, 2004], is to take the *greatest homomorphism*
  - ▶ Yields  $x \mapsto x \cdot X^\omega, y \mapsto y \cdot X^\omega$

# This talk

We develop the following in a coalgebraic setting:

- ▶ Games
- ▶ Strategies
- ▶ Strategy outcomes
  - ▶ Limit-based definition
  - ▶ Largest homomorphism approach
- ▶ Game outcomes (the “strategy semantics”)
  - ▶ Limit-based definition

## Defining a game

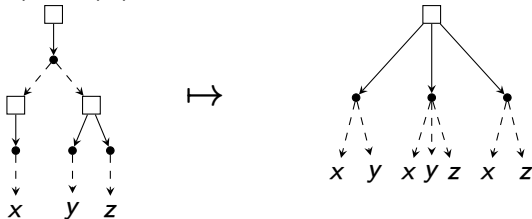
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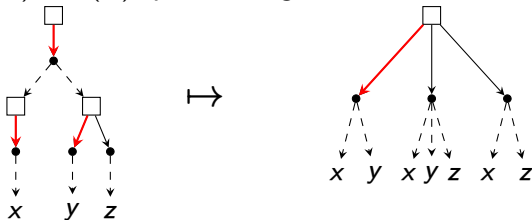
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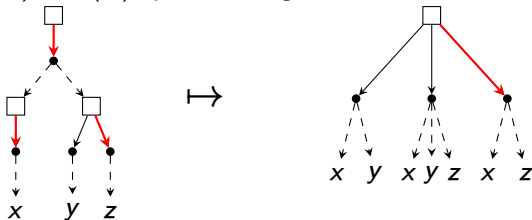
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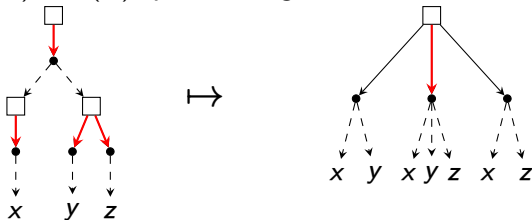
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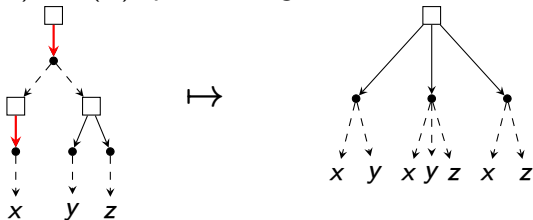
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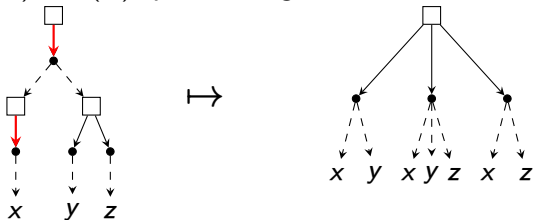
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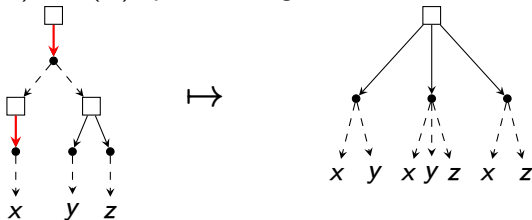
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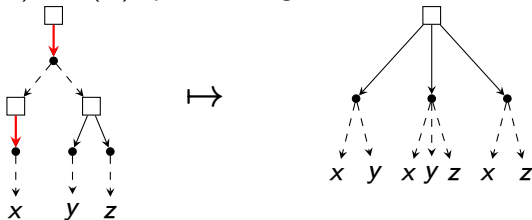


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- ▶ Use  $\dashv\vdash$  to denote a **KI**( $\mathcal{G}$ ) map
  - ▶  $\gamma : X \dashv\vdash F(X)$

## Defining a strategy

- ▶ Consider a chain in **Rel**:

$$1 \xrightarrow{\sigma_*} X \xrightarrow{\sigma_0} X^2 \xrightarrow{\sigma_1} X^3 \xrightarrow{\sigma_2} \dots$$

- ▶  $\sigma_*(*) \subseteq X$  gives the starting states of the strategy

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- ▶  $S_n \subseteq X^{n+1}$  with  $\text{Im}(\sigma_n) = S_{n+1}$

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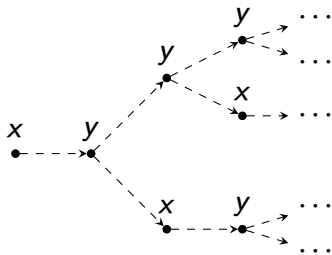
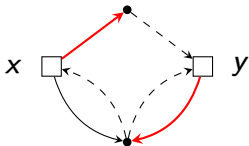
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- ▶ We can use  $(-)^{\dagger} : \mathbf{Rel}_{\text{rt+sep}} \cong \mathbf{Set}$  to obtain a cochain  $S_0 \leftarrow S_1 \leftarrow \dots$  in **Set**
  - ▶ Strategy outcome: take the limit of this cochain

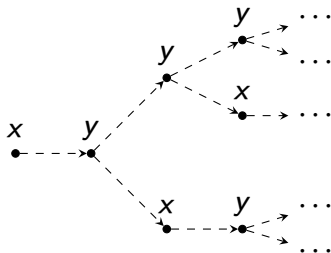
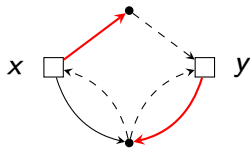
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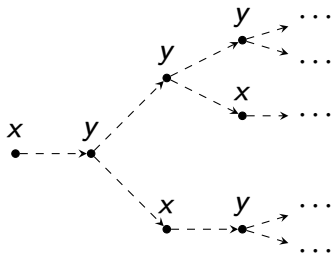
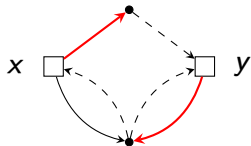
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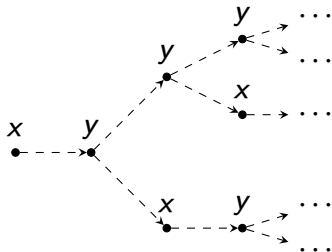
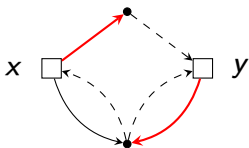
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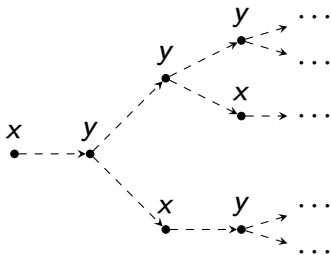
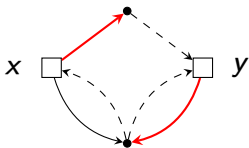


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$$\alpha^\omega + \alpha^* y^\omega$$

where  $\alpha := xyy^*$

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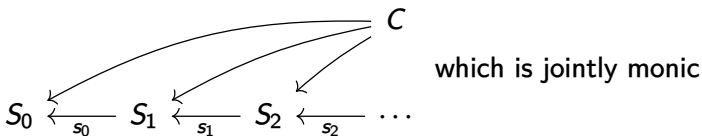
- ▶  $h(u)$  is the set of executions which complete  $u$  according to  $\sigma$ 
  - ▶ Think of this as a greatest fixed point computation
  - ▶ Start with  $h(u) = Z$ , then refine backwards according to  $\sigma$

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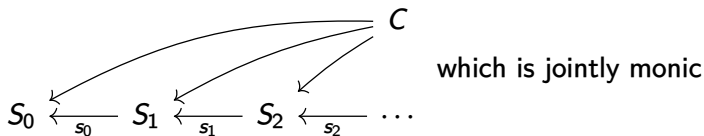


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- ▶ The greatest homomorphism is identified with the *largest jointly monic cone*:
  - ▶ This is precisely the *limiting cone*
  - ▶ Our two notions of outcome are the same

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- ▶ Define  $\gamma_n : X \dashrightarrow F^n(X)$  by iterating  $\gamma : X \dashrightarrow F(X)$  in  $\mathbf{KI}(\mathcal{G})$
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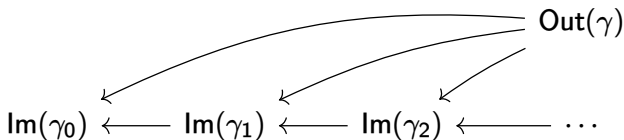
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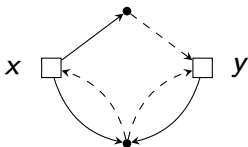
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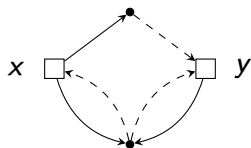


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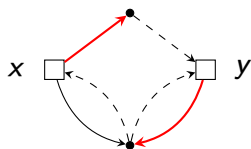
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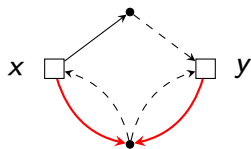
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- ▶ Future work:
  - ▶ Probabilistic opponents (MDPs, SSGs)
  - ▶ Weighted opponents (energy games)
  - ▶ Parity/Büchi games [Urabe et al., 2016]
  - ▶ Product construction of game with an automaton

# References I



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