Traces via Strategies

(Games via Coalgebra)

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Outline

- Games
- Traces
- Representing games coalgebraically
- Strategies
- Traces via strategies

Motivation: controller synthesis

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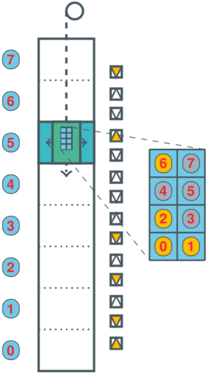
- We have specification (as a logical formula).
- Synthesis question: is there are a controller strategy which every play satisfies the specification?
- Example "every request is served" (a liveness property)



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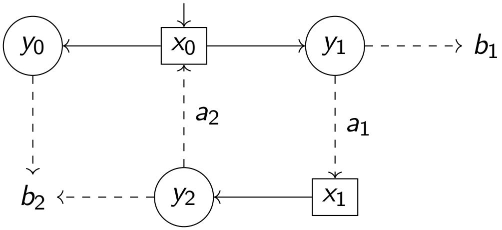

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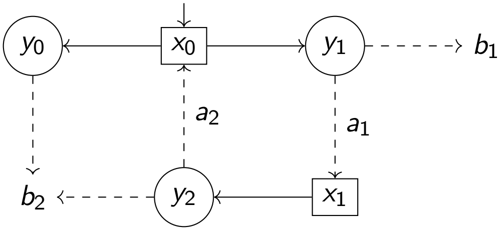

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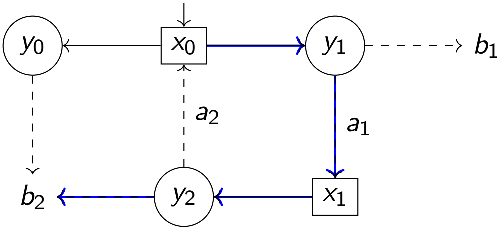

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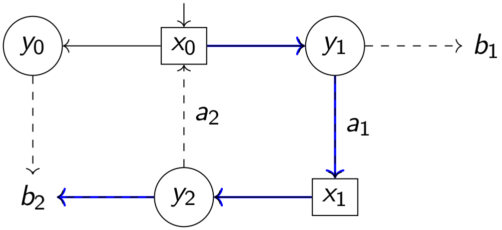

- A play is a sequence of states and observations, arising from controller and environment moves, ending in a terminating observation. e.g. $x_0 y_1 a_1 x_1 y_2 b_2$
- A strategy is a partial function which extends partial plays, it must be defined over all partial plays which conform to it.

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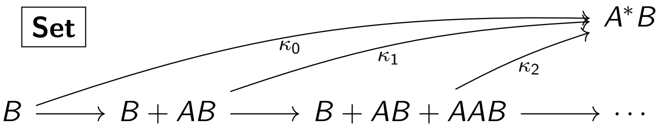
P is a monad with $\mathbf{Kl}(P) \cong \mathbf{Rel}$

Traces, coalgebraically

A^*B is the *initial algebra* for the functor $B + A(-) : \mathbf{Set} \to \mathbf{Set}$



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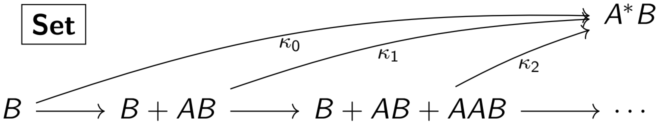
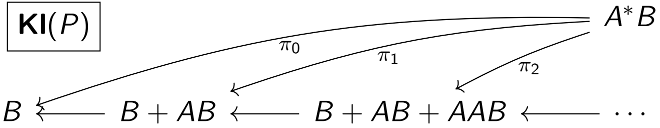


General categorical machinery allows us to lift this chain to the category of relations, and reverse the arrows[1]:



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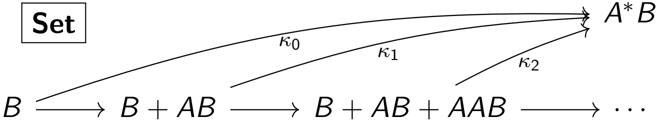

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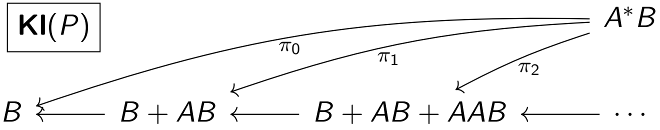

Thus A^*B is a *final coalgebra* in the category of relations!

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Traces by coinduction

For every LTS $c : X \to P(B + A \times X)$, there is a *unique coalgebra morphism* into A^*B.

$$\boxed{\textbf{Rel}} \qquad \begin{array}{ccc} X & \dashrightarrow & A^*B \\ \downarrow c & & \downarrow \wr \\ B + A \times X & \dashrightarrow & B + A \times A^*B \end{array}$$

This dashed morphism in **Rel** is a function $X \to P(A^*B)$ which assigns each state to it's set of traces!

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- With $c^* : X \to PH_X(X)$ defined as the composite

$$(X \xrightarrow{\langle \mathrm{id}, c \rangle} X \times P(B + A \times X) \xrightarrow{\mathrm{stl}} P(X \times (B + A \times X))$$

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- And it follows from a general coalgebraic result that:

$$X \xrightarrow{\mathrm{exec}_c} (XA)^* XB \xrightarrow{f_{\pi_2}} A^* B \quad \text{where } \pi_2 : H_X(Y) \to H(Y).$$

$$X \xrightarrow{\mathrm{tr}_c} A^* B$$

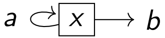
Rel

Recap

Because the monad P has lots of nice properties, we automatically get trace/execution maps:

$$X \xrightarrow{\text{exec}_c} (XA)^* XB \xrightarrow{f_{\pi_2}} A^* B, \qquad \text{tr}_c : X \to A^* B$$

Rel

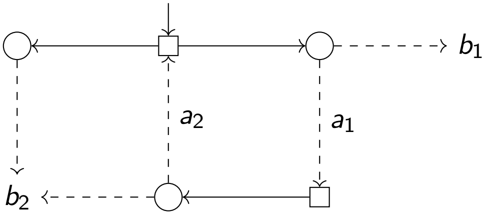


$$\text{exec}_c(x) = \{xb, xaxb, xaxaxb, xaxaxaxb, \ldots\}$$

$$\text{tr}_c(x) = \{b, ab, aab, aaab, \ldots\}$$

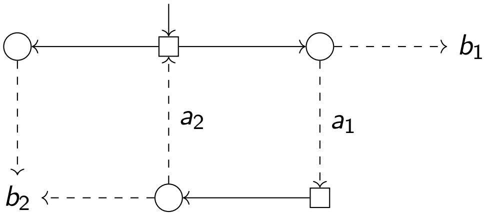
Games

Recall:



How do we turn this into a function $X \to M(HX)$?
i.e. Which monad M do we choose?

Finding the monad



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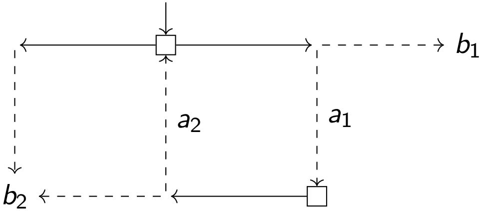

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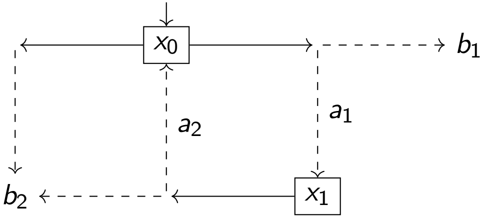

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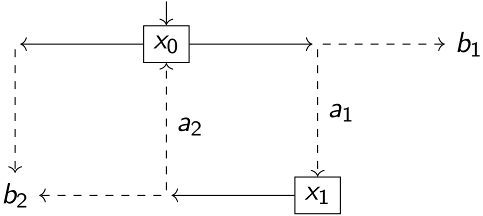

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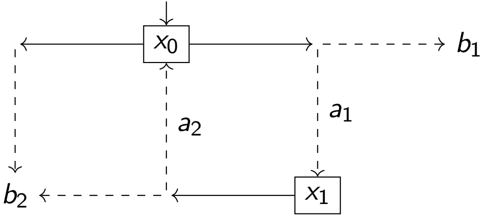

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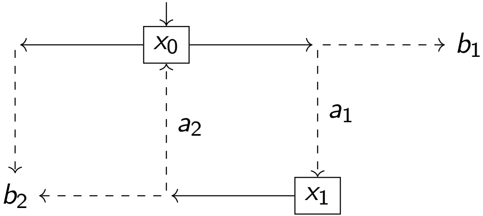

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Is PP a monad?

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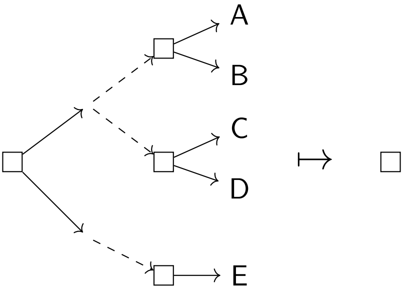
Let $A, B, C, D, E \subseteq X$



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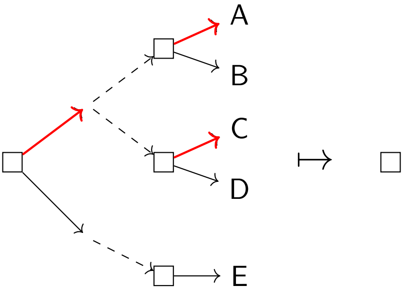

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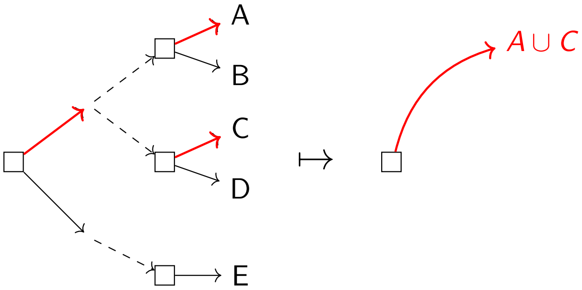

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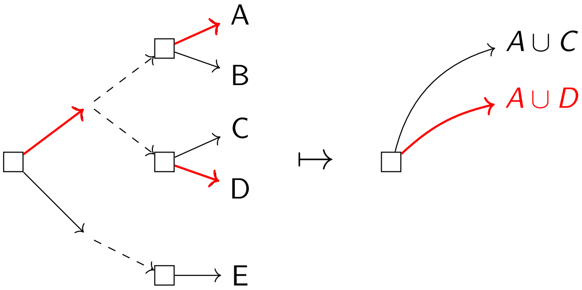

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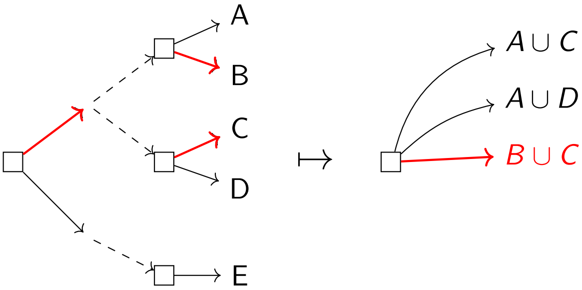

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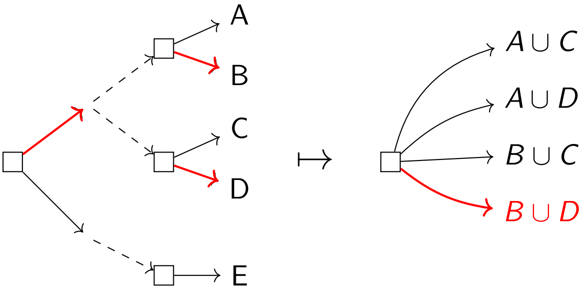

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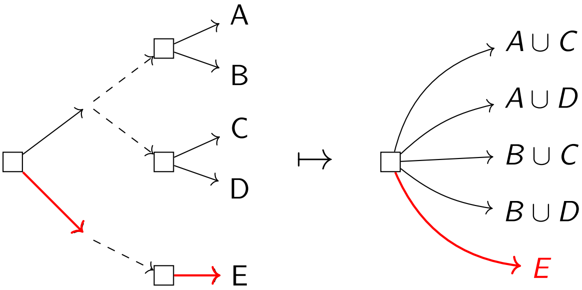

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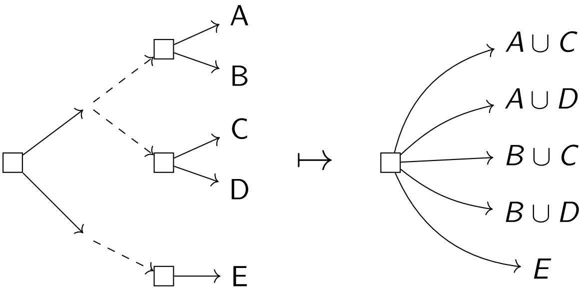

$$\{\{\{A, B\}, \{C, D\}\}, \{\{E\}\}\} \mapsto \{A \cup C, A \cup D, B \cup C, B \cup D, E\}$$

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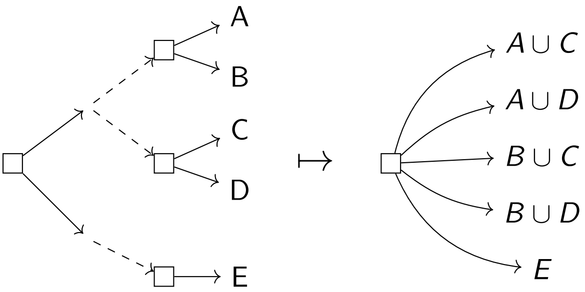

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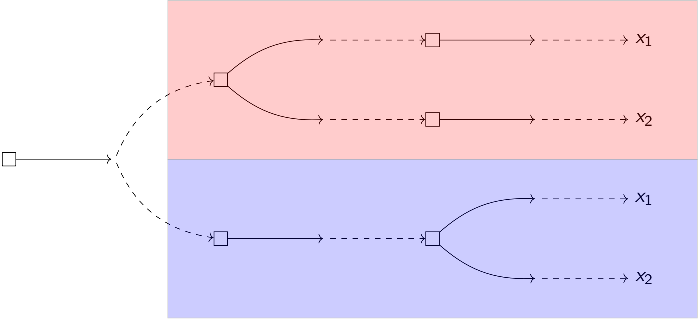
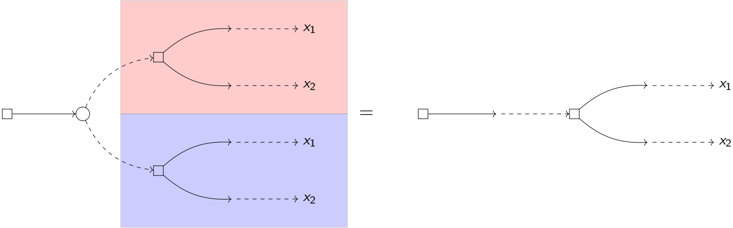
$$\Upsilon \in PPPP(X) \mapsto \{\bigcup \mathrm{Im}(f) \mid \exists \upsilon \in \Upsilon,\ f : \upsilon \xrightarrow{*} PP(X)\}$$

where $f : \upsilon \xrightarrow{*} PP(X)$ is a *choice function*: $\forall \mathcal{U} \in \upsilon : \mathcal{U} \in f(\mathcal{U})$.

Failure of associativity





A monad structure on PP is a "no-go" [KS18]

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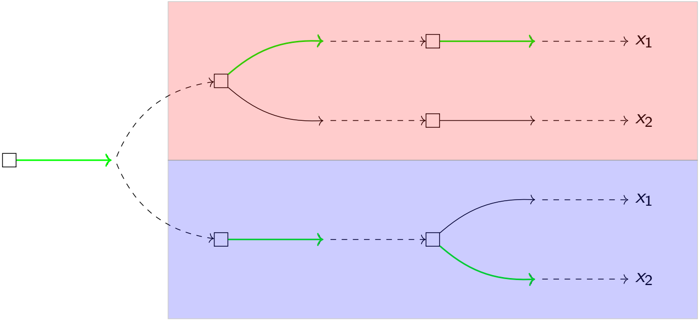

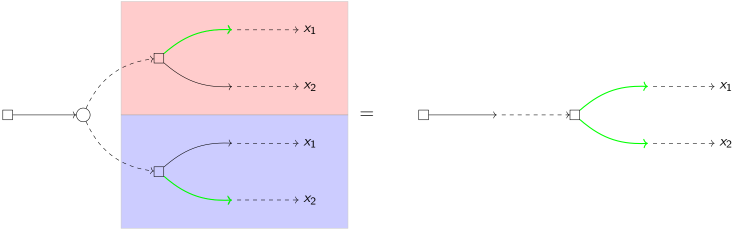

(Weak) Distributive laws

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- Given two monads (S, μ^T, η^T) and (T, μ^T, η^T), a *distributive law* $\delta : TS \to ST$ is a natural transformation satisfying some coherence conditions involving $\mu^T, \mu^S, \eta^T, \eta^S$.

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- A *weak distributive law* $\delta : TS \to ST$ only satisfies the diagrams involving μ^T, μ^S, η^S.

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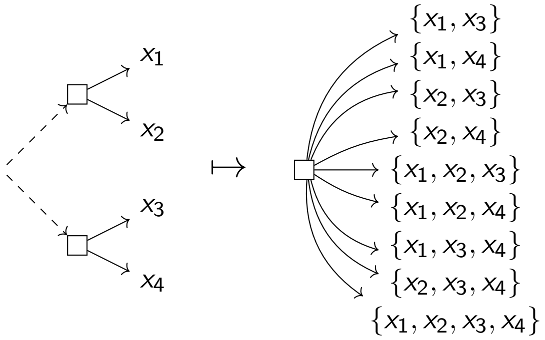
A weak distributive law $\delta : PP \to PP$

- Gives us a way of swapping environment-then-controller branching into controller-then-environment.



$$\delta(\{U_i\}_{i \in I}) = \{\bigcup_{i \in I} V_i \mid V_i \subseteq^+ U_i \text{ for all } i \in I\}$$

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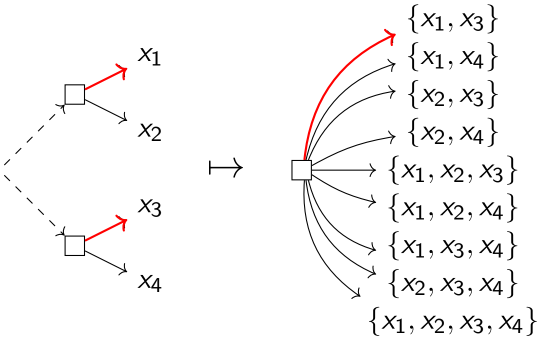

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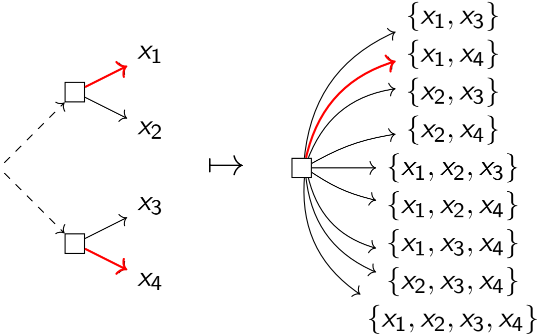

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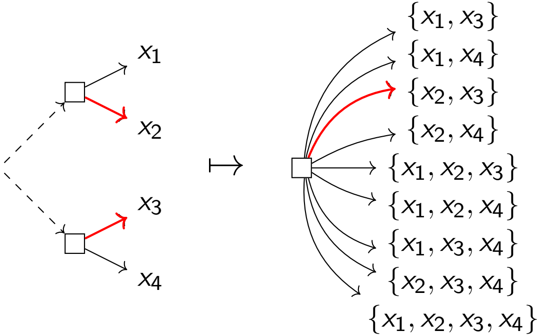

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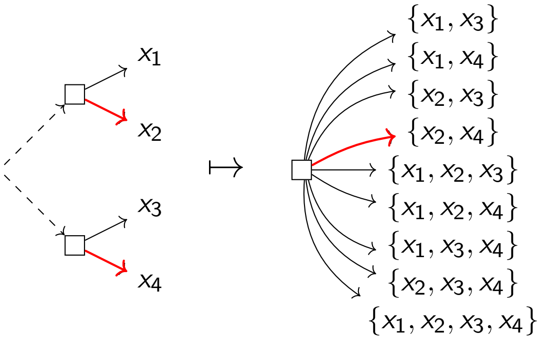

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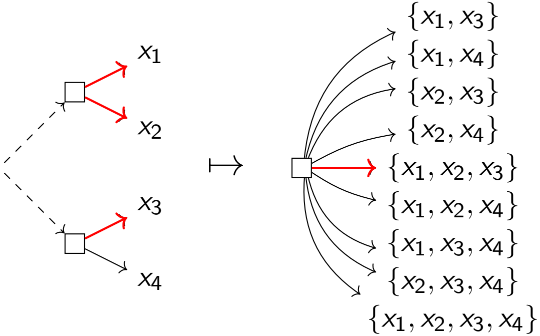

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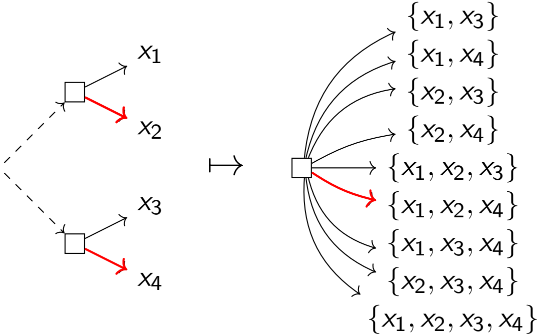

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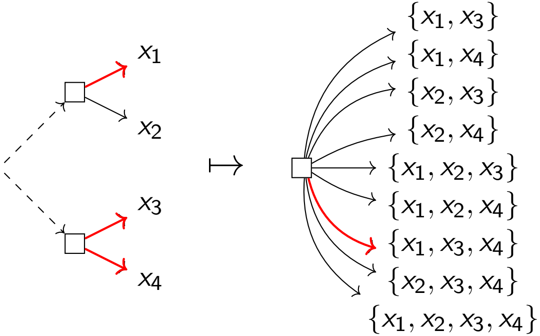

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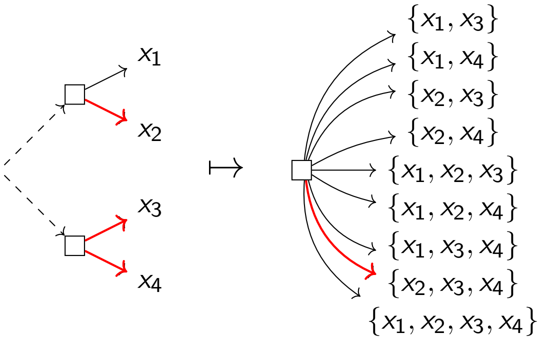

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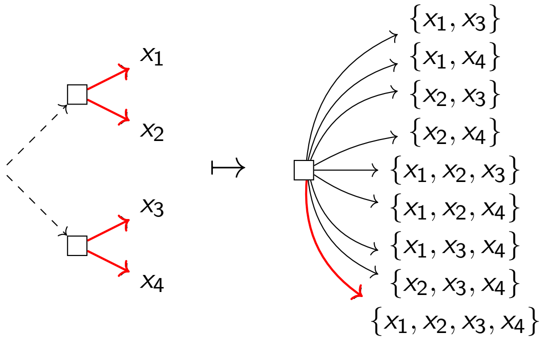

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Trace semantics

- We can build a monad

$$\widetilde{PP}(X) = \{\mathcal{U} \subseteq X \mid \mathcal{U} \text{ is closed under arbitrary union}\}$$

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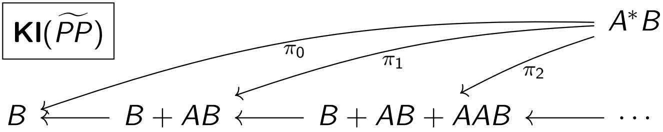
- Recall: General categorical machinery allows us to lift this chain to the category of relations, and reverse the arrows:



with various assumptions on $\widetilde{PP}$

Refining the monad

-
-
-
-

These fixes are discussed in [PC25] (to appear)

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$$Q(X) = \{U \subseteq^{+}_{\omega} X\}$$

$$\widetilde{PQ}(X) = \{\mathcal{U} \subseteq Q(X) \mid \mathcal{U} \text{ is closed under binary union}\}$$

$$\delta : QP \to PQ$$

$$\delta(\{U_1, \ldots, U_n\}) := \{V_1 \cup \cdots \cup V_n \mid V_i \subseteq^{+}_{\omega} U\}$$

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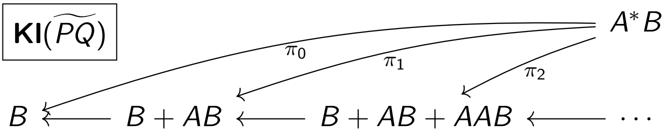
Traces and Executions

A^*B is the final $B + A(-)$-coalgebra in $\mathbf{Kl}(\widetilde{PQ})$.



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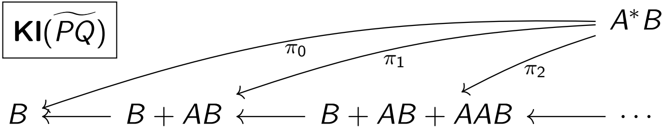
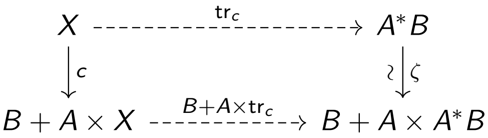


Thus we have trace and execution maps by coinduction:



$$\mathrm{tr}_c : X \to \widetilde{PQ}(A^*B)$$

Traces and Executions

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$\mathbf{Kl}(\widetilde{PQ})$

$$B \leftarrow B + AB \leftarrow B + AB + AAB \leftarrow \cdots$$

$$\pi_0 : A^*B \to B, \quad \pi_1 : A^*B \to B + AB, \quad \pi_2 : A^*B \to B + AB + AAB$$

Thus we have trace and execution maps by coinduction:

$$\begin{array}{ccc}
X & \xrightarrow{\ \mathrm{exec}_c\ } & (XA)^*XB \\
\downarrow{\scriptstyle c^*} & & \wr\downarrow{\scriptstyle \zeta} \\
X \times (B + A \times X) & \xrightarrow{\ B+A\times\mathrm{exec}_c\ } & X \times (B + A \times (XA)^*XB)
\end{array}$$

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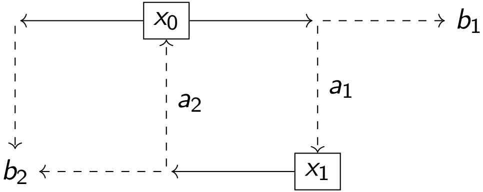
What is $\mathrm{tr}_c(x_0)$?



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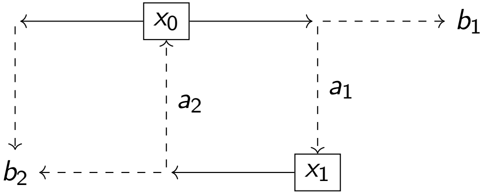

$$\{b_1\}$$

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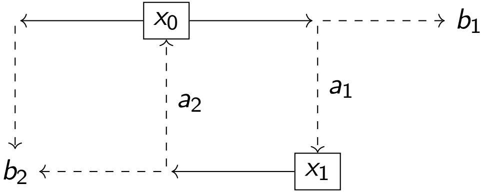

$\{b_1\}$ No

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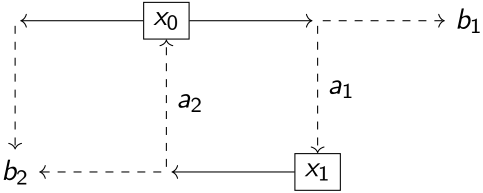

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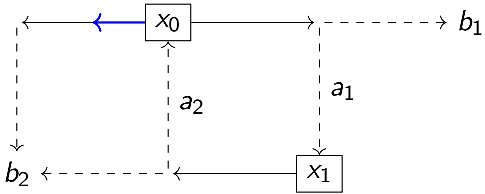

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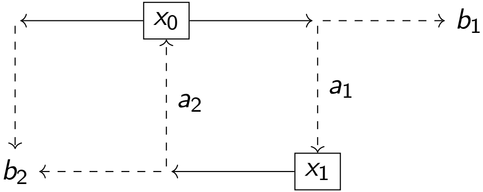

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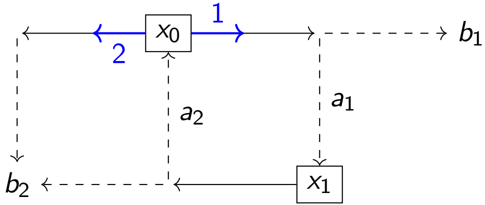

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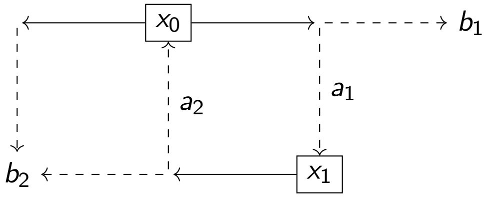

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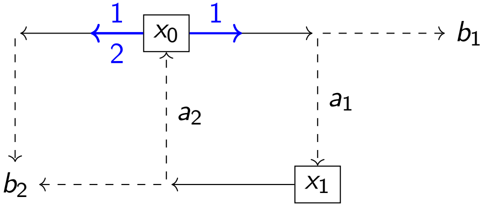

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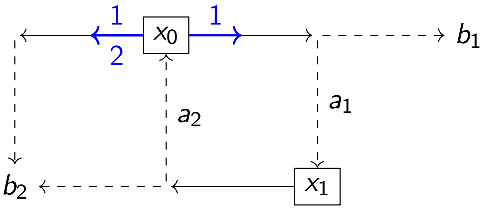

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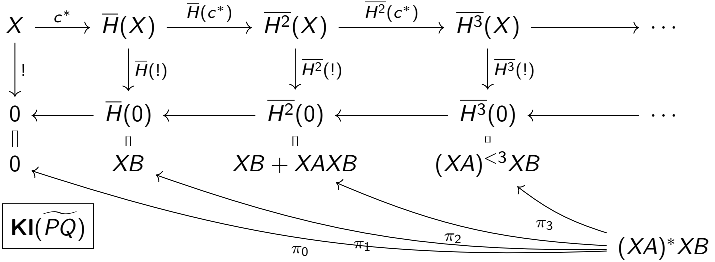
(Theorem sketch) for all $U \subseteq A^* B$:
$U \in \mathrm{tr}_c(x) \implies$ there is a strategy which enforces U
$U \in \mathrm{tr}_c(x) \impliedby^*$ there is a strategy which enforces U
*almost

Executions on the final sequence



where $\overline{H} : \mathbf{Kl}(\widetilde{PQ}) \to \mathbf{Kl}(\widetilde{PQ})$ is the lifting of $X \times (B + A \times (-))$

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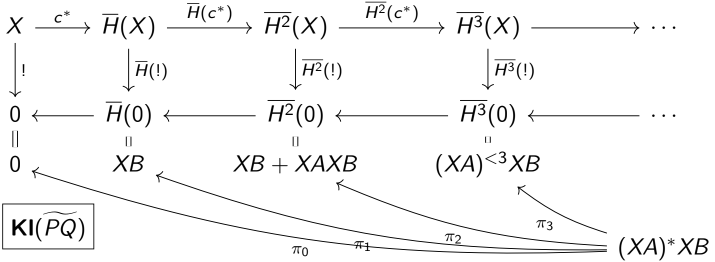

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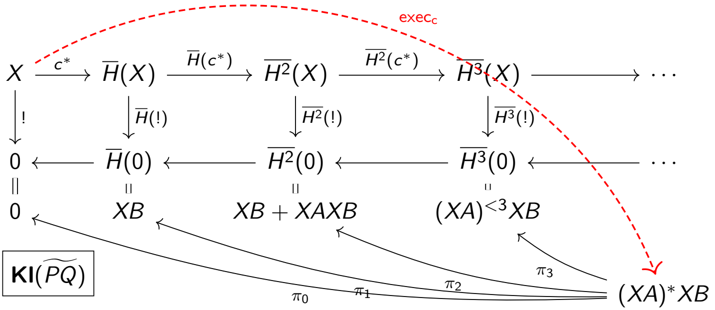

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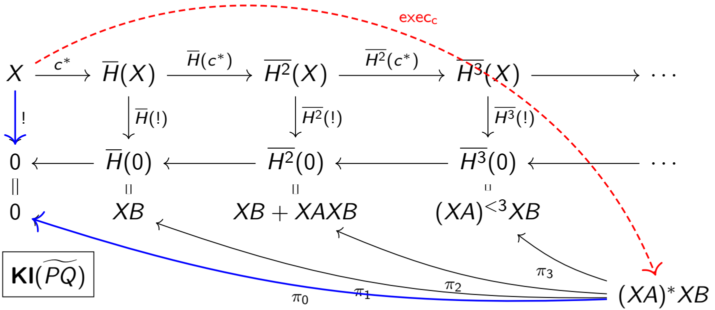

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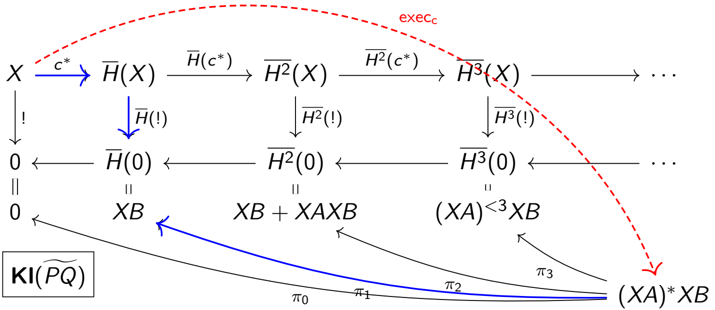

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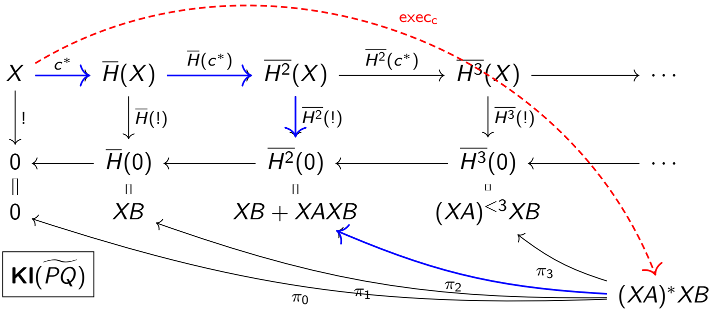

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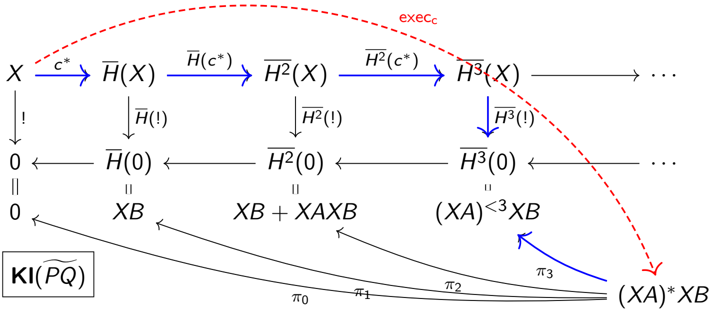

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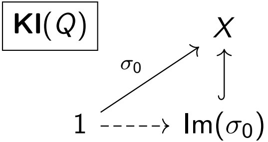
Where are the strategies?

- $\sigma_0 : 1 \to Q(X)$ will pick an initial state



Where are the strategies?

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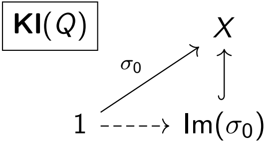

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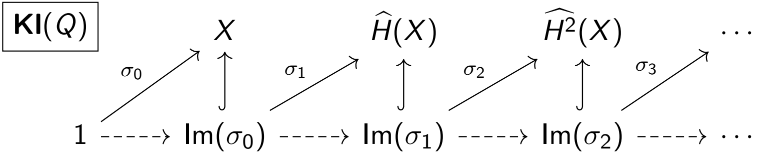

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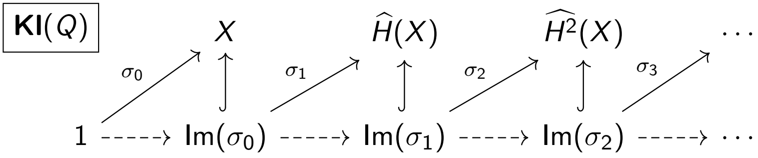
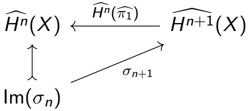
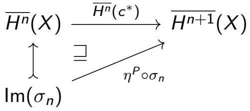


(left) σ extends partial plays, (right) we choose a successor in c:





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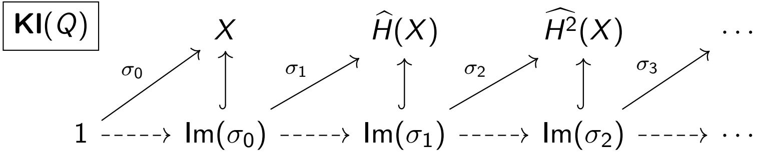

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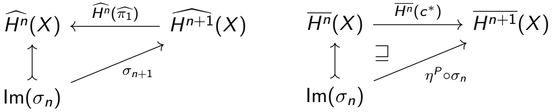


The n-depth plays comes from composition:

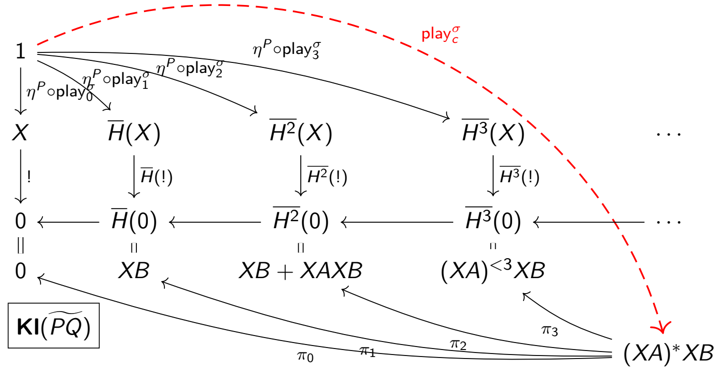
$$\mathrm{play}_n^{\sigma} = (1 \dashrightarrow \mathrm{Im}(\sigma_0) \dashrightarrow \cdots \dashrightarrow \mathrm{Im}(\sigma_n) \rightarrowtail H^n(X))$$

Play outcomes

To define the play outcome, first lift a strategy into $\mathbf{Kl}(\widetilde{PQ})$

$$\eta^P \circ \sigma_n : \mathrm{Im}(\sigma_n) \to \widetilde{PQ} H^{n+1}(X)$$

Then we can reuse that $(XA)^*XB$ is the limit of the final sequence:



Main theorem

Theorem

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Lemma

$$c_n^*(x) = \{\mathrm{play}_n^\sigma \mid \sigma \text{ starts in } x\}$$

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- What do we gain from doing this coalgebraically?
  - Replace Q with the finite distribution monad D!
  - Generic coinductive algorithms for strategy synthesis.

Conclusion

- Towards strategy synthesis...
  - Product construction?
  - General theorem about memoryless strategies?
  - Infinite traces, continuous probability monads?
- An axiomatic presentation.
- Simple stochastic games?

Bibliography I

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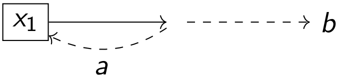

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